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Multi-node higher order expansions of a function ☆

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Abstract

By using the values and higher derivatives of a function at the given nodes, a kind of multi-node higher order expansion of the function is presented. The error terms of the expansions are given. Particular examples are the extensions of the Taylor polynomials, Bernstein polynomials and Lagrange interpolation polynomials. The expansions are numerical approximation polynomials and very useful particular for the functions for which the higher derivatives can be obtained easily.

Keywords: Numerical approximation; Functional expansion; Asymptotic error expansion; Bernstein polynomial; Interpolation polynomial

1. Introduction

It is well known that the usual Bernstein polynomials are defined on the interval $[0, 1]$ as follows:

$$B_n(f,x) := \sum_{i=0}^{n} B_{ni}(x) f\left(\frac{i}{n}\right), \tag{1}$$

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where $B_{ni}(x) = C_n^i x^i (1-x)^{n-i}$. In the book [4], it is shown that, if $f \in C^{2r}[0,1]$, $r \geqslant 1$, then

$$f(x) - B_n(f,x) = -\sum_{j=1}^{2r} \frac{T_{nj}(x)}{j!} f^{(j)}(x) + o\left(\frac{1}{n^r}\right), \tag{2}$$

where

$$T_{nj}(x) = \sum_{i=0}^{n} B_{ni}(x)\left(\frac{i}{n} - x\right)^j.$$

Under the same conditions, the following asymptotic expansion can be found in [5]:

$$\begin{aligned} f(x) - B_n(f,x) = &-\sum_{j=1}^{2r} \frac{T_{nj}(x)}{j!} f^{(j)}(x) + O\left[\frac{x(1-x)}{n^r}\left(x(1-x) + \frac{1}{n}\right)^{r-1} \right. \\ &\left. \times\, \omega\left(f^{(2r)}, \sqrt{\frac{x(1-x)}{n} + \frac{1}{n}}\right)\right], \end{aligned} \tag{3}$$

where $\omega(f,\delta)$ is the modulus of continuity of f.

In order to get (3), the following lemma is used.

Lemma 1.

$$|T_{n,2r}(x)| \leqslant M_1 \frac{x(1-x)}{n^r}\left(x(1-x) + \frac{1}{n}\right)^{r-1},$$

$$|T_{n,2r+1}(x)| \leqslant M_2 \frac{x(1-x)}{n^{r+1}}\left(x(1-x) + \frac{1}{n}\right)^{r-1},$$

where M_1 and M_2 are independent of n and x.

For the Bernstein polynomials defined on a simplex, some error bounds and some kind of asymptotic error expansion are given in [3]. Some asymptotic expansion formulas are also discussed in [2].

The Bernstein polynomials are functional approximation polynomials using only function values. If the higher derivatives of a function are used, we can expect a higher order expansion similar to (2) or (3). The Taylor expansion is a common expansion using higher derivatives of a function. However, it is worthwhile to discuss how to properly use higher derivatives of a function to construct an expansion. In this paper, using the values and higher derivatives of a function at the given nodes, we construct higher order expansions of a function. The expansions are numerical approximation polynomials and are obtained by using special properties of the operator and a particular choice of the expansion coefficients.

The present paper is organized as follows. In Section 2, based on a class of basis functions, a kind of higher order expansion and its error terms are shown. In

Section 3, based on the Bernstein basis functions and the Lagrange basis functions, the expansions and their error terms are shown. The comparisons of the errors of the given cubic approximation polynomials and the cubic Hermite interpolation polynomial are detailed.

2. Multi-node higher order expansions

Let

$$L(f,x) := \sum_{i=0}^{n} \varphi_i(x) f(x_i) \tag{4}$$

be a linear operator in $C[a,b]$, with the nodes $a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = b$ and $\varphi_i \in C[a,b]$ chosen so that L reproduces all polynomials of degree $\leqslant m$. Then, in particular,

$$\sum_{i=0}^{n} \varphi_i(x)(x_i - t)^m = (x-t)^m, \quad x \in [a,b], \quad t \in R. \tag{5}$$

This is equivalent to

$$\sum_{i=0}^{n} \varphi_i(x)(x - x_i)^k = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \ldots, m. \end{cases}$$

Thus, replacing $f(x_i)$ in (4) by the Taylor expansion $\sum_{j=0}^{r} (x - x_i)^j f^{(j)}(x_i)/j!$ gives an operator which reproduces all polynomials of degree $\leqslant \max\{m, r\}$, but this does not reproduce all polynomials of degree $\leqslant m + r$. For that, as we will show, it suffices to replace $f(x_i)$ in (4) by $\sum_{j=0}^{r} a_j (x - x_i)^j f^{(j)}(x_i)/j!$, with

$$a_j := \frac{r!(m + r - j)!}{(m + r)!(r - j)!}, \quad j = 0, 1, \ldots, r. \tag{6}$$

In other words, we claim that, for this choice of the a_j, the following multi-node higher expansion

$$H_{nr}(f,x) := \sum_{i=0}^{n} \varphi_i(x) \sum_{j=0}^{r} \frac{a_j}{j!}(x - x_i)^j f^{(j)}(x_i) \tag{7}$$

reproduces all polynomials of degree $\leqslant m + r$.

Obviously, if we take $m = 0$ and $x_0 = x_1 = \cdots = x_n$, then (7) is the Taylor expansion polynomial at one point due to $\sum_{i=0}^{n} \varphi_i(x) = 1$.

For the analysis of $H_{nr}(f,x)$, we need the following two lemmas concerning the a_j.

Lemma 2. *Let k, m, r be nonnegative integers, $p = \min\{k, r\}$, then*

$$\sum_{j=0}^{p} (-1)^j \frac{(m+r-j)!}{j!(k-j)!(r-j)!} = \frac{m!}{k!r!} \prod_{j=1}^{r} (m+j-k). \tag{8}$$

Proof. Comparing the coefficients of x^r on the two sides of the equality $(1+x)^{k-m-1} = (1+x)^k(1+x)^{-m-1}$, we have

$$\frac{(k-m-1)(k-m-2)\cdots(k-m-r)}{r!} = \sum_{j=0}^{p} \frac{k!}{j!(k-j)!} \cdot \frac{(-m-1)(-m-2)\cdots(-m-r+1)}{(r-j)!}.$$

From this we get (8). □

Lemma 3. *Let k, m, r be nonnegative integers, $p = \min\{k, r\}$, $a_0 = 1$. Then, for $j = 1, 2, \ldots, r$, (6) are the unique solutions of the following linear equations*:

$$\sum_{j=0}^{p} (-1)^{k-j} \frac{a_j}{j!(k-j)!} = 0, \quad k = m+1, \; m+2, \ldots, m+r. \tag{9}$$

Proof. From (8), we can deduce inductively that the coefficient matrix of (9) is nonsingular. (8) also means that a_j $(j = 1, 2, \ldots, r)$ given in (6) are the solutions of the linear equations (9). □

Now we give the error terms of expansion (7).

Theorem 1. *Let $f(x) \in C^{m+r+1}[a, b]$, $x \in [a, b]$, then for (7), we have*

$$f(x) - H_{nr}(f, x) = \frac{1}{(m+r)!} \sum_{i=0}^{n} \varphi_i(x) \int_{x_i}^{x} (x-t)^r (x_i - t)^m f^{(m+r+1)}(t)\, dt. \tag{10}$$

Proof. For $x \in [a, b]$, by Taylor expansion, we have

$$\begin{aligned} &\sum_{j=0}^{r} \frac{a_j}{j!}(x - x_i)^j f^{(j)}(x_i) \\ &\quad = \sum_{j=0}^{r} \frac{a_j}{j!}(x - x_i)^j \left[\sum_{k=j}^{m+r} \frac{1}{(k-j)!}(x_i - x)^{k-j} f^{(k)}(x) \right. \\ &\qquad \left. + \frac{1}{(m+r-j)!} \int_{x}^{x_i} (x_i - t)^{m+r-j} f^{(m+r+1)}(t)\, dt \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m+r} \left[\sum_{j=0}^{p} (-1)^{k-j} \frac{a_j}{j!(k-j)!} \right] (x-x_i)^k f^{(k)}(x) \\
&\quad + \sum_{j=0}^{r} \frac{a_j}{j!(m+r-j)!} (x-x_i)^j \int_x^{x_i} (x_i - t)^{m+r-j} f^{(m+r+1)}(t)\, dt,
\end{aligned}
$$

where $p = \min\{k, r\}$. By the restriction of φ_i and (9), we get

$$
\begin{aligned}
H_{nr}(f,x) &= \sum_{i=0}^{n} \varphi_i(x) \sum_{j=0}^{r} \frac{a_j}{j!}(x-x_i)^j f^{(j)}(x_i) \\
&= f(x) + \frac{1}{(m+r)!} \sum_{i=0}^{n} \varphi_i(x) \sum_{j=0}^{r} \frac{r!}{j!(r-j)!}(x-x_i)^j \\
&\quad \times \int_x^{x_i} (x_i - t)^{m+r-j} f^{(m+r+1)}(t)\, dt \\
&= f(x) - \frac{1}{(m+r)!} \sum_{i=0}^{n} \varphi_i(x) \int_{x_i}^{x} (x-t)^r (x_i - t)^m f^{(m+r+1)}(t)\, dt.
\end{aligned}
$$

This means (10) holds. □

Theorem 2. *Let $f(x) \in C^{m+r+1}[a,b]$, $x \in [a,b]$, then for* (7), *we have*

$$
f(x) - H_{nr}(f,x) = \frac{1}{(m+r)!} \int_a^b K_{mr}(x,t) f^{(m+r+1)}(t)\, dt, \tag{11}
$$

where

$$
K_{mr}(x,t) = (x-t)^r \left[(x-t)_+^m - \sum_{i=0}^{n} \varphi_i(x)(x_i - t)_+^m \right]. \tag{12}
$$

Proof. Since

$$
\begin{aligned}
&\int_a^b (x-t)^r [(x-t)_+^m - (x_i - t)_+^m] f^{(m+r+1)}(t)\, dt \\
&\quad = \int_a^x (x-t)^r [(x-t)^m - (x_i - t)^m] f^{(m+r+1)}(t)\, dt \\
&\qquad - \int_x^{x_i} (x-t)^r (x_i - t)^m f^{(m+r+1)}(t)\, dt,
\end{aligned}
$$

$$
\sum_{i=0}^{n} \varphi_i(x)[(x-t)^m - (x_i - t)^m] = 0,
$$

we have

$$\sum_{i=0}^{n} \varphi_i(x) \int_a^b (x-t)^r [(x-t)_+^m - (x_i-t)_+^m] f^{(m+r+1)}(t)\,dt$$
$$= \sum_{i=0}^{n} \varphi_i(x) \int_{x_i}^x (x-t)^r (x_i-t)^m f^{(m+r+1)}(t)\,dt. \tag{13}$$

From (10) and (12), we get (11). □

Theorem 3. *Let $f \in C^{m+r+1}[a,b]$, $r \geqslant 0$, $x \in [a,b]$. If $K_{m0}(x,t)$ is of one sign in $[a,b]$, then when r is an even number,*

$$f(x) - H_{nr}(f,x) = \frac{(-1)^m m!r!}{(m+r)!(m+r+1)!} f^{(m+r+1)}(\xi) \sum_{i=0}^{n} \varphi_i(x)(x-x_i)^{m+r+1}, \tag{14}$$

for some $\xi \in [a,b]$, and when r is an odd number,

$$f(x) - H_{nr}(f,x) = (-1)^m \frac{m!r!}{(m+r)!(m+r+1)!} \times \left[f^{(m+r+1)}(\xi_1) \sum_{i=0}^{k} \varphi_i(x)(x-x_i)^{m+r+1} + f^{(m+r+1)}(\xi_2) \sum_{i=k+1}^{n} \varphi_i(x)(x-x_i)^{m+r+1} \right], \tag{15}$$

where $x \in [x_k, x_{k+1}]$, $0 \leqslant k \leqslant n-1$, for some $\xi_1 \in [a,x]$, $\xi_2 \in [x,b]$.

Proof. Since $K_{m0}(x,t)$ is of one sign in $[a,b]$, then $K_{mr}(x,t) = (x-t)^r K_{m0}(x,t)$ is of one sign in $[a,b]$ when r is an even number and $K_{mr}(x,t)$ is of one sign in $[a,x]$ or $[x,b]$ when r is an odd number. Therefore, when r is an even number, according to (11), we have

$$f(t) - H_{nr}(f,x) = \frac{1}{(m+r)!} f^{(m+r+1)}(\xi) \int_a^b K_{mr}(x,t)\,dt$$

for some $\xi \in [a,b]$. When r is an odd number, we have

$$f(t) - H_{nr}(f,x) = \frac{1}{(m+r)!} \left[f^{(m+r+1)}(\xi_1) \int_a^x K_{mr}(x,t)\,dt + f^{(m+r+1)}(\xi_2) \int_x^b K_{mr}(x,t)\,dt \right]$$

for some $\xi_1 \in [a,x]$, $\xi_2 \in [x,b]$.

According to (13) for the special case $f^{(m+r+1)}(t) = 1$, we have

$$\int_a^b K_{mr}(x,t)\,dt = \sum_{i=0}^{n} \varphi_i(x) \int_{x_i}^{x} (x-t)^r (x_i - t)^m\,dt$$
$$= (-1)^m \frac{m!r!}{(m+r+1)!} \sum_{i=0}^{n} \varphi_i(x)(x-x_i)^{m+r+1}.$$

From this we get (14).

For $x \in [x_k, x_{k+1}]$, $0 \leqslant k \leqslant n-1$, since

$$\int_a^x K_{mr}(x,t)\,dt = \int_a^b K_{mr}(x,t)(x-t)_+^0\,dt,$$

the same formula (13), with the special case $f^{(m+r+1)}(t) = (x-t)_+^0$, immediately gives

$$\int_a^x K_{mr}(x,t)\,dt = (-1)^m \frac{m!r!}{(m+r+1)!} \sum_{i=0}^{k} \varphi_i(x)(x-x_i)^{m+r+1}.$$

Then, we have

$$\int_x^b K_{mr}(x,t)\,dt = \int_a^b K_{mr}(x,t)\,dt - \int_a^x K_{mr}(x,t)\,dt$$
$$= (-1)^m \frac{m!r!}{(m+r+1)!} \sum_{i=k+1}^{n} \varphi_i(x)(x-x_i)^{m+r+1}.$$

Thus, we get (15) immediately. □

3. On two special cases

3.1. The case of the Bernstein basis functions

In formula (7), we take uniform node sequence $x_i = a + ih$, $i = 0, 1, \ldots, n$, $h = (b-a)/n$ and the Bernstein basis functions as follows

$$\varphi_i(x) = B_{ni}(x) = C_n^i \left(\frac{x-a}{b-a}\right)^i \left(\frac{b-x}{b-a}\right)^{n-i}, \quad i = 0, 1, \ldots, n.$$

Then $\sum_{i=0}^{n} B_{ni}(x) = 1$, $\sum_{i=0}^{n} B_{ni}(x)(x-x_i) = 0$. Therefore, with $m = 1$, (7) for this case can be written as follows

$$H_{nr}(f,x) = B_n(f,x) + \frac{1}{r+1} \sum_{j=1}^{r} \frac{r+1-j}{j!} \sum_{i=0}^{n} B_{ni}(x)(x-x_i)^j f^{(j)}(x_i), \tag{16}$$

where $B_n(f,x) = \sum_{i=0}^{n} B_{ni}(x)f(x_i)$ is the Bernstein polynomial defined on $[a,b]$. Formula (16) is an extension of the Bernstein polynomial. For $t \in [a,b]$, since

$$K_{10}(x,t) = (x-t)_+ - \sum_{i=0}^{n} B_{ni}(x)(x_i - t)_+ \leqslant 0,$$

Theorem 3 is applicable for this case. As an illustration, we have the following.

Theorem 4. *Let* $f(x) \in C^4[a,b]$, $x \in [a,b]$, $x_0 = a$, $x_1 = b$, *then for* (16) *we have*

$$|f(x) - H_{12}(f,x)| \leqslant \tfrac{1}{864}(b-a)^4 \|f^{(4)}\|, \tag{17}$$

where $\|f^{(4)}\| = \max_{a \leqslant x \leqslant b} |f^{(4)}|$.

Proof. Let $x = a + (b-a)t$, then $0 \leqslant t \leqslant 1$. According to Theorem 3, with $n = 1$, $m = 1$, $r = 2$, $x_0 = a$, $x_1 = b$, and $\varphi_i(x) = B_{1i}(x)(i = 0,1)$, we have

$$f(x) - H_{12}(f,x) = -\frac{2!}{3!4!} f^{(4)}(\xi)[B_{10}(x)(x-a)^4 + B_{11}(x)(x-b)^4],$$

for some $\xi = \xi(x) \in [a,b]$, therefore

$$\begin{aligned} |f(x) - H_{12}(f,x)| &\leqslant \tfrac{1}{72}(b-a)^4(1-t)t[1-3(1-t)t]\|f^{(4)}\| \\ &\leqslant \tfrac{1}{864}(b-a)^4\|f^{(4)}\|. \qquad \square \end{aligned}$$

Remark 1. $H_{12}(f,x)$ is a cubic polynomial. Comparing it with the cubic Hermite interpolation polynomial $H_3(f,x)$, which uses the same information about f, namely $f(a), f'(a), f(b), f'(b)$, we have

$$|f(x) - H_3(f,x)| \leqslant \tfrac{1}{384}(b-a)^4\|f^{(4)}\| = \tfrac{9}{4} \times \tfrac{1}{864}(b-a)^4\|f^{(4)}\|.$$

This means that $H_{12}(f,x)$ has a better error bound than $H_3(f,x)$.

Similar to the asymptotic expansion (3), we have the following results.

Theorem 5. *Let* $x \in [a,b]$, $R(f,x) = f(x) - H_{nr}(f,x)$, *then when* $f \in C^{r+1}[a,b]$ *and* $r \geqslant 1$, *we have*

$$\begin{aligned} |R(f,x)| \leqslant A \frac{(x-a)(b-x)}{n^{\frac{r+1}{2}}} &\left[(x-a)(b-x) + \frac{(b-a)^2}{n}\right]^{\frac{r-1}{2}} \\ &\times \omega\left[f^{(r+1)}, \sqrt{\frac{(x-a)(b-x)}{n}} + \frac{b-a}{n}\right]. \end{aligned} \tag{18}$$

When $f \in C^r[a,b]$ and $r \geqslant 2$, we have

$$|R(f,x)| \leqslant B\frac{(x-a)(b-x)}{n^{\frac{r}{2}}}\left[(x-a)(b-x)+\frac{(b-a)^2}{n}\right]^{\frac{r-2}{2}} \times \omega\left[f^{(r)}, \sqrt{\frac{(x-a)(b-x)}{n}}+\frac{b-a}{n}\right], \tag{19}$$

where A and B are independent of n and x.

Proof. By the Taylor formula and (9), we have

$$\begin{aligned}\sum_{j=0}^{r}\frac{a_j}{j!}(x-x_i)^j f^{(j)}(x_i) &= \sum_{j=0}^{r}\frac{a_j}{j!}(x-x_i)^j \\ &\quad \times\left[\sum_{k=j}^{r}\frac{1}{(k-j)!}(x_i-x)^{k-j}f^{(k)}(x)\right. \\ &\quad \left.+\frac{1}{(r+1-j)!}(x_i-x)^{r+1-j}f^{(r+1)}(\xi_{ij})\right] \\ &= f(x)-\frac{1}{r+1}(x-x_i)f'(x) \\ &\quad +\sum_{j=0}^{r}(-1)^{r+1-j}\frac{a_j}{j!(r+1-j)!}(x-x_i)^{r+1} \\ &\quad \times(f^{(r+1)}(\xi_{ij})-f^{(r+1)}(x)),\end{aligned}$$

where ξ_{ij} is a number between x_i and x. Therefore

$$\begin{aligned}R(f,x) &= f(x)-H_{nr}(f,x) \\ &= \frac{1}{r+1}\sum_{i=0}^{n}\sum_{j=0}^{r}(-1)^{r+1-j}\frac{1}{j!(r-j)!}B_{ni}(x)(x-x_i)^{r+1} \\ &\quad \times(f^{(r+1)}(x)-f^{(r+1)}(\xi_{ij})).\end{aligned}$$

When $|x-x_i| \leqslant \sqrt{\frac{(x-a)(b-x)}{n}}+\frac{b-a}{n}$,

$$|f^{(r+1)}(x)-f^{(r+1)}(\xi_{ij})| \leqslant \omega(\cdot) := \omega\left(f^{(r+1)}, \sqrt{\frac{(x-a)(b-x)}{n}}+\frac{b-a}{n}\right).$$

When $|x-x_i| > \sqrt{\frac{(x-a)(b-x)}{n}}+\frac{b-a}{n}$,

$$\begin{aligned}|f^{(r+1)}(x)-f^{(r+1)}(\xi_{ij})| &\leqslant \omega(f^{r+1}, |x-x_i|) \\ &\leqslant \left[1+|x-x_i|\Big/\left(\sqrt{\frac{(x-a)(b-x)}{n}}+\frac{b-a}{n}\right)\right]\omega(\cdot)\end{aligned}$$

$$\leqslant \frac{2n}{\sqrt{n(x-a)(b-x)}+b-a}|x-x_i|\omega(\cdot)$$
$$\leqslant \frac{2n}{\sqrt{n(x-a)(b-x)+(b-a)^2}}|x-x_i|\omega(\cdot).$$

Let $x=a+t(b-a)$, then $t\in[0,1]$, $x-x_i=(b-a)\left(t-\frac{i}{n}\right)$. By Lemma 1 and Hölder inequality, we have

$$\left(\sum_{i=0}^{n} B_{ni}(x)|x-x_i|^{r+1}\right)^2 \leqslant \left(\sum_{i=0}^{n} B_{ni}(x)(x-x_i)^{2r}\right)\left(\sum_{i=0}^{n} B_{ni}(x)(x-x_i)^2\right)$$
$$\leqslant M\frac{(x-a)(b-x)}{n^r}\left[(x-a)(b-x)+\frac{(b-a)^2}{n}\right]^{r-1} \times \frac{(x-a)(b-x)}{n},$$

where M is independent of n and x. Therefore, for $r\geqslant 1$, we have

$$\sum_{i=0}^{n} B_{ni}(x)|x-x_i|^{r+1} \leqslant \sqrt{M}\frac{(x-a)(b-x)}{n^{\frac{r+1}{2}}}\left[(x-a)(b-x)+\frac{(b-a)^2}{n}\right]^{\frac{r-1}{2}}.$$

Let

$$I_1=\left\{i\colon |x-x_i|\leqslant\sqrt{\frac{(x-a)(b-x)}{n}}+\frac{b-a}{n}\right\},$$
$$I_2=\left\{i\colon |x-x_i|>\sqrt{\frac{(x-a)(b-x)}{n}}+\frac{b-a}{n}\right\}.$$

Sum up, we have

$$|R(f,x)|\leqslant \frac{1}{r+1}\left[\sum_{i\in I_1}\sum_{j=0}^{r}\frac{1}{j!(r-j)!}B_{ni}(x)|x-x_i|^{r+1}+\sum_{i\in I_2}\sum_{j=0}^{r}\frac{1}{j!(r-j)!} \times B_{ni}(x)|x-x_i|^{r+2}\frac{2n}{\sqrt{n(x-a)(b-x)+(b-a)^2}}\right]\omega(\cdot)$$
$$\leqslant \frac{2^r}{(r+1)!}A_1\frac{(x-a)(b-x)}{n^{\frac{r+1}{2}}}\left[(x-a)(b-x)+\frac{(b-a)^2}{n}\right]^{\frac{r-1}{2}}\omega(\cdot),$$

where A_1 is independent of n and x. The proof of (18) is completed. In the same way, the proof of (19) can be completed. □

3.2. The case of the Lagrange basis functions

In formula (7), we take

$$\varphi_i(x) = l_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \quad i = 0, 1, \ldots, n.$$

Hence, $L(f, x) = L_n(f, x)$, the Lagrange interpolation polynomial of degree $\leqslant n$, well known to reproduce all polynomials of degree $\leqslant n$. Therefore, with $m = n$, (7) for this case can be written as follows:

$$H_{nr}(f, x) = L_n(f, x) + \sum_{j=1}^{r} \frac{a_j}{j!} \sum_{i=0}^{n} l_i(x)(x - x_i)^j f^{(j)}(x_i). \tag{20}$$

Formula (20) is an extension of the Lagrange interpolation polynomial. Further, $K_{n0}(x, t)$ is of one sign (as a function of t) in $[a, b]$ since it is the error at x of the polynomial interpolant to $(\cdot - t)_+^n$, hence can be written $(x - x_0)(x - x_1)\cdots(x - x_n)[x_0, x_1, \ldots, x_n, x](\cdot - t)_+^n$ while, as a function of t, $[x_0, x_1, \ldots, x_n, x](\cdot - t)_+^n$ is a B-spline with knots $x_0, x_1, \ldots, x_n, x$, hence of one sign (see, e.g., [1]). Therefore, Theorem 3 applies in this case. As an illustration, we have the following.

Theorem 6. *Let $f(x) \in C^4[a, b]$, $x \in [a, b]$, $x_0 = a$, $x_1 = (a + b)/2$, $x_2 = b$, then for* (20), *we have*

$$|f(x) - H_{21}(f, x)| \leqslant \tfrac{0.19}{288}(b - a)^4 ||f^{(4)}||, \tag{21}$$

where $||f^{(4)}|| = \max_{a \leqslant x \leqslant b} |f^{(4)}|$.

Proof. We might as well let $x \in [x_0, x_1]$, $x = x_1 + \frac{b-a}{2}t$, $-1 \leqslant t \leqslant 0$. According to Theorem 3, with $n = 2$, $m = 2$, $r = 1$, $x_0 = a$, $x_1 = (a + b)/2$, $x_2 = b$, and $\varphi_i(x) = l_i(x)$ $(i = 0, 1, 2)$, we have

$$f(x) - H_{21}(f, x) = \frac{2!}{3!4!}\left\{ f^{(4)}(\xi_1) l_0(x)(x - a)^4 + f^{(4)}(\xi_2)\left[l_1(x)\left(x - \frac{a + b}{2}\right)^4 + l_2(x)(x - b)^4 \right] \right\},$$

for some $\xi_1 \in [a, x]$, $\xi_2 \in [x, b]$, therefore

$$\begin{aligned} |f(x) - H_{21}(f, x)| &\leqslant \tfrac{1}{1152}(b - a)^4 (t^3 - t)(t^3 + 3t^2 + 1)||f^{(4)}|| \\ &\leqslant \tfrac{0.76}{1152}(b - a)^4 ||f^{(4)}|| = \tfrac{0.19}{288}(b - a)^4 ||f^{(4)}||. \end{aligned}$$

In the same way, the proof of (21) can be completed when $x \in [x_1, x_2]$. □

Remark 2. $H_{21}(f,x)$ is a cubic polynomial. Comparing it with the cubic Hermite interpolation polynomial $H_3(f,x)$, we have

$$|f(x)-H_3(f,x)|\leqslant\tfrac{1}{384}(b-a)^4||f^{(4)}||=\tfrac{3}{0.76}\times\tfrac{0.19}{288}(b-a)^4||f^{(4)}||.$$

This means that $H_{21}(f,x)$ has a better error bound than $H_3(f,x)$. On the other hand, $H_{21}(f,x)$ uses more information about f than $H_3(f,x)$ does, namely the value of f and of f' at $(a+b)/2$.

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